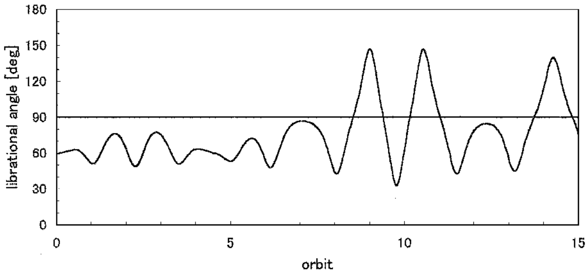

Fig. 4 Variation of librational angle.

The equation of infinitesimal additional motion around the steady-state solution can be obtained as follows:

$$\ddot{\tilde{\theta}}_i + (p_i\omega)^2\tilde{\theta}_i = 0 \tag{5}$$

where

$$p_i = \sqrt{3 - \left(C_i^2/3\right)} \tag{6}$$

and

$$\theta = \theta_{0i} + \tilde{\theta}_i \tag{7}$$

as shown in Fig. 3.

Forced Resonance Due to Atmospheric Density Variation

It is easy to see that forced resonance may occur when the angular velocity of the subsatellite libration is equal to a multiple of the orbital angular velocity as follows:

$$p_i\omega = n\omega \qquad (n = 1, 2, 3, \ldots) \tag{8}$$

Substituting Eqs. (2), (3), and (6) into Eq. (8), one can arrive at an equation describing the condition of the forced resonance due to the atmospheric density variation as

$$\frac{C_D A}{m\ell} = \frac{2}{\rho_i(R_E + h)^2}\sqrt{9 - 3n^2} \tag{9}$$

Equation (9) has real solution only in the case where $n = 1$, that is, the tethered subsatellite librates once per orbit. Thus, Eq. (9) yields

$$\frac{C_D A}{m\ell} = \frac{2\sqrt{6}}{\rho_i(R_E + h)^2} \tag{10}$$

where the left-hand side of Eq. (10) consists of the parameters dependent on the form of the TSS; on the other hand, the right-hand side of Eq. (10) is a function of the altitude h.

Numerical Example

Suppose R_E, h, and ρ_2 to be 6378 km, 160 km, and 1.04×10^{-9} kg/m^3, respectively; then the value of $2\sqrt{6}/[\rho_2(R_E+h)^2]$ is obtained to be 1.10×10^{-4}. If parameters C_D, A, m, and ℓ are selected as 2.2, 1.0 m^2, 20 kg, and 1000 m, respectively, then the value of $C_D A/m\ell$ is also equal to 1.10×10^{-4}. Now, Eq. (10) is satisfied, and the tethered subsatellite may be forced to resonate by the atmospheric density variation.

Figure 4 shows the time history of the libration θ by using Eq. (1) with the initial conditions: $\Psi = -\Psi_1$, $\theta = \theta_{01}$, and $\dot{\theta} = 0$. It can be seen that the libration of the tethered subsatellite diverges and crosses the local horizontal in nine orbits.

Conclusions

It is concluded that the librational motion of the tethered subsatellite during the stationkeeping phase is forced to resonate by the atmospheric density variation. The numerical result shows that the libration diverges and crosses the local horizontal in nine orbits when the angular velocity of the TSS's libration is equal to the orbital angular velocity. This result also indicates a possibility that the librational motion during the deployment and retrieval phases may be induced to diverge by the external perturbation due to the atmospheric density variation.

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Worst-Case Control for Discrete-Time Uncertain Nonlinear Systems

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Introduction

ONE of the important requirements for a control system is the so-called robustness. During the last decade, robust control of uncertain dynamic systems has been extensively studied. Considerable attention has been devoted to the problems of robust stabilization and robust performance of uncertain dynamic systems in both the continuous and discrete-time contexts.[1,2]

We extend the control design methodology proposed in Ref. 3 to handle the problem of robust control of a class of uncertain discrete-time systems. The methodology is extended in two ways. The first idea is to allow parametric uncertainty in the controlled output equation. The second aspect is that the nonlinear uncertainty contains system state, control input, and disturbance input variables all together. The class of uncertain systems is described by a state-space model with linear nominal parts and nonlinear time-varying norm-bounded parameter uncertainty in the state and output equations. Attention is focused on the design of linear state feedback controllers. We address the problem of robust H_∞ control in which both robust stability and a prescribed H_∞ performance are required to be achieved irrespective of the uncertainties. Our results show that the described problem can be converted to a robust H_∞ control for a related discrete-time system with time-varying norm-bounded linear uncertainty. Therefore, the Riccati inequality approach[4] can be used to obtain a solution to the problem of robust H_∞ control of nonlinear uncertain systems.

Robust Control Result

Consider the following uncertain system:

$$(\Sigma): \qquad x_{k+1} = Ax_k + B_1 w_k + B_2 u_k + \Delta_1(x_k, w_k, u_k)$$

$$x_0 = 0 \tag{1}$$

$$z_k = Cx_k + D_1 w_k + D_2 u_k + \Delta_2(x_k, w_k, u_k) \tag{2}$$

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where $x_k \in \Re^n$ represents the state, $w_k \in \Re^p$ is the disturbance input that belongs to $\ell_2[0, \infty]$, $u_k \in \Re^m$ is the control input, and $z_k \in \Re^r$ is the controlled output; A, B_1, B_2, C, D_1, and D_2 are known real constant matrices of appropriate dimensions that describe the nominal system, and $\Delta_1(x_k, w_k, u_k)$ and $\Delta_2(x_k, w_k, u_k)$ are real time-varying matrices representing norm-bounded parameter uncertainties.

The admissible parameter uncertainties are of the form

$$\|\Delta_i(x_k, w_k, u_k)\| \le a_i\|x_k\| + b_i\|w_k\| + c_i\|u_k\| \qquad i = 1, 2 \tag{3}$$

$$\le a\|x_k\| + b\|w_k\| + c\|u_k\| \tag{4}$$

$$\forall x_k \in \Re^n, \qquad w_k \in \Re^p, \qquad u_k \in \Re^m, \qquad k \in \mathbf{Z}$$

where $a = \max\{a_1, a_2\} \ge 0$, $b = \max\{b_1, b_2\} \ge 0$, and $c = \max\{c_1, c_2\} \ge 0$ are known constant numbers, and $\mathbf{Z}$ is a set of integer numbers.

We denote the corresponding uncertainty set by

$$\Omega_i(x_k, w_k, u_k) = \{\Delta_i(x_k, w_k, u_k) : \|\Delta_i(x_k, w_k, u_k)\| \le a_i\|x_k\| + b_i\|w_k\| + c_i\|u_k\|\}, \qquad i = 1, 2$$

We are concerned with the design of a robust linear state feedback control law $u_k \in U = \{-Kx_k\}$, K being the feedback gain, in $\ell_2[0, \infty]$ for Eqs. (1) and (2). We investigate the design of a state feedback controller $(\mathcal{G})$ for Eqs. (1) and (2) that reduces $z = \{z_k, k = 1, 2, \ldots\}$ uniformly for any $w = \{w_k, k = 1, 2, \ldots\}$ in the sense that, given a scalar $\gamma > 0$, the closed-loop system of Eqs. (1) and (2) with the controller $(\mathcal{G})$ satisfies

$$\|z\|_{[0,\infty]} < \gamma\|w\|_{[0,\infty]} \tag{5}$$

for any nonzero $w \in \ell_2[0, \infty]$, $k \in \mathbf{Z}$ and for all admissible uncertainties. For this situation, the closed-loop system of Eqs. (1) and (2) with $(\mathcal{G})$ is said to have a robust H_∞ performance γ over the horizon $[0, \infty]$.

The robust H_∞ control problem is then as follows: Given a scalar $\gamma > 0$, design a linear state feedback controller $\mathcal{G}$ based on the state x_k, such that the closed-loop system (1) and (2) with $\mathcal{G}$ is robustly stable and has a robust H_∞ performance over $[0, \infty]$.

We assume, without loss of generality, that $\gamma = 1$. Therefore, we rescale B_1 by γ^{-1}, replacing B_1 in Eq. (1) by $B_\gamma = \gamma^{-1}B_1$. We also need the assumption that (A, B_2) is stabilizable.

Theorem: Consider system (1) and (2) satisfying inequality (4). If there exist a scalar $\alpha > 0$ and a matrix $P > 0$ satisfying

$$\theta_1 \triangleq I - B_\gamma^T P B_\gamma - D_1^T D_1 - B_0 > 0 \tag{6}$$

$$A^T P A - P + C^T C + A_0 + \tilde{B}_1^T \theta_1^{-1} \tilde{B}_1 - \theta_3^T \theta_2^{-1} \theta_3 < 0 \tag{7}$$

where

$$\theta_2 \triangleq B_2^T P B_2 + D_2^T D_2 + C_0 + \tilde{B}_3^T \theta_1^{-1} \tilde{B}_3$$

$$\theta_3 \triangleq \tilde{B}_2 + \tilde{B}_3^T \theta_1^{-1} \tilde{B}_1 \tag{8}$$

$$\tilde{B}_1 \triangleq B_\gamma^T P A + D_1^T C, \qquad \tilde{B}_2 \triangleq B_2^T P A + D_2^T C$$

$$\tilde{B}_3 \triangleq B_\gamma^T P B_2 + D_1^T D_2$$

$$A_0 \triangleq (a + b + c)\alpha A^T P^2 A + (ab + ac)\alpha P^2 + a^2 P + (a + b + c)\alpha C^T C + [a^2 + (6a/\alpha) + (ab + ac)\alpha]I$$

$$B_0 \triangleq (a + b + c)\alpha B_\gamma^T P^2 B_\gamma + bc\alpha P^2 + b^2 P + (a + b + c)\alpha D_1^T D_1 + \{b^2 + [(2ab + 6b)/\alpha] + bc\alpha\}I$$

$$C_0 \triangleq (a + b + c)\alpha B_2^T P^2 B_2 + c^2 P^2 + (a + b + c)\alpha D_2^T D_2 + [(2ac + 2bc + 6c)/\alpha + c^2]I \tag{9}$$

then the control law

$$u_k = F x_k = -\theta_2^{-1}\theta_3 x_k \tag{10}$$

achieves robust global asymptotic stability with robust unitary disturbance attenuation

$$\|z\| < \|w\|$$

for all nonzero $w \in \ell_2[0, \infty]$ and all $\Delta_i(x_k, w_k, u_k) \in \Omega_i(x_k, w_k, u_k)$, $i = 1, 2$.

Proof: The proof of this theorem is given in Ref. 5.

Remark: In the proof of the theorem, for the sake of simplicity, we use only one scaling parameter α for all quadratic form inequalities. Of course, we can select different α for each inequality, which will result in Riccati inequality (7) and the controller (10) to contain more scaling parameters. Furthermore, the existence of the parameter $\alpha > 0$ in the theorem can be checked using convex optimization over Linear Matrix Inequalities (LMIs).

Application Example

A linearized model of aircraft in longitudinal motion at an altitude of 27,000 ft and airspeed of 610 ft/s (Ref. 6), subjected to a gust impact is shown to satisfy the uncertain system (1) and (2), where

$$A = \begin{bmatrix} 0.9991 & 0.0007 & 0 & -0.0013 \\ -0.0070 & 0.9511 & 0.0233 & 0 \\ 0.0102 & -0.7800 & 0.9084 & 0 \\ 0.0001 & -0.0100 & 0.0239 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.9991 & 0.0007 & 0 \\ -0.0070 & 0.9511 & 0.0233 \\ 0.0102 & -0.7800 & 0.9084 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.0007 & 0.9394 & -0.4542 & 0 \\ 0.0007 & 0.9511 & 0 & 0 \\ 0.9991 & 0.0007 & 0 & -0.0013 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.0007 & 0.9394 & -0.4542 \\ 0.0007 & 0.9511 & 0 \\ 0.9991 & 0.0007 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$B_2 = [0 - 0.0171 \quad -1.1251 \quad -0.0143]^T$$

$$D_2 = [0.5454 \quad -0.0171 \quad 0 \quad 0]^T$$

$$\|\Delta_i(x_k, w_k, u_k)\| \le 1.2\|x_k\| + 0.8\|w_k\| + 1.2\|u_k\|, \qquad i = 1, 2$$

$$\Delta_l(x_k, w_k, u_k) = 1.2 f_{1k} E_1 x_k + 0.8 f_{2k} E_2 w_k + 1.2 f_{3k} E_3 u_k$$

where $f_{ik} \in \Re^4$ and $\|f_{ik}\| \le 1$, $i = 1, 2, 3$, and

$$E_1 = [1 \quad 0 \quad 0 \quad 0], \qquad E_2 = [0 \quad 1 \quad 0], \qquad E_3 = 1$$

The system state is $x_k = [V_x/U_0 \;\; \alpha \;\; q \;\; \theta]^T$, where V_x/U_0 is the normalized horizontal airspeed, α is the angle of attack, q is the pitch rate, θ is the pitch angle, and U_0 is the steady-state airspeed. Note that the preceding system is obtained under sampled data measurement of $\Delta t = 0.025$ s. The gust variables are $w_k = [u_g \;\; \alpha_g \;\; q_g]^T$, where u_g, α_g, and q_g are the horizontal and vertical translational velocity of turbulence and the inertial pitching velocity of perturbation, respectively. Note that the turbulence affecting the pitch angle is indirectly computed from the pitch rate. The control surface of the system u is the deflection angle of the elevator. The nonlinear uncertainties of the system are due to the coupling between the state variables and gust variables. This coupling is caused by the aeroelasticity of the aircraft and control surfaces during gust impact. As long as the gust impact is under a certain value, maximum bending and torsional moments will be under structural allowance. This situation is related to the critical gust load problem. In this example, we show that the proposed method can be used to handle the effects of system uncertainties and to alleviate the gust impact.

If we choose

$$Q = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \qquad \varepsilon = 1, \qquad \gamma = 4$$

then Eq. (7) has a positive definite solution

$$P = \begin{bmatrix} 0.1360 & -0.0778 & 0.0382 & 0.2197 \\ -0.0778 & 0.5041 & -0.0930 & -0.5838 \\ 0.0382 & -0.0930 & 0.1488 & 0.3122 \\ 0.2197 & -0.5838 & 0.3122 & 0.9639 \end{bmatrix}$$

satisfying $\theta_1 > 0$, and the controller (10) is given by

$$u(t) = 0.2395(V_x/U_0) - 0.3166\alpha + 0.0880q + 0.4605\theta$$

Using this controller, the simulation results are shown in Figs. 1–4 with initial condition $x = [0 \;\; 5.73\,\text{deg} \;\; 5.73\,\text{deg/sampled-data(s)} \;\; 0]^T$.

Note that the horizontal axes of Figs. 1–4 have been transferred to the sampled-data time based on seconds instead of integers. In Fig. 1, the solid lines represent the vertical velocity and pitch rate using robust control, and the dashed lines are without control. It is shown that the amplitudes and the length of time of the short-period modes to reach the steady state for the controlled system are smaller and less than those of the uncontrolled states. This indicates that the controlled system has lower torsion and bending moments than that of the uncontrolled system. Similarly, the phugoid modes (the horizontal velocity and the pitch angle) for controlled and uncontrolled systems are shown in Figs. 2 and 3. Again, Figs. 2 and 3 demonstrate that the long period motion of aircraft decays very well using the robust controller. Finally, the robust state feedback controller trajectory is presented in Fig. 4.

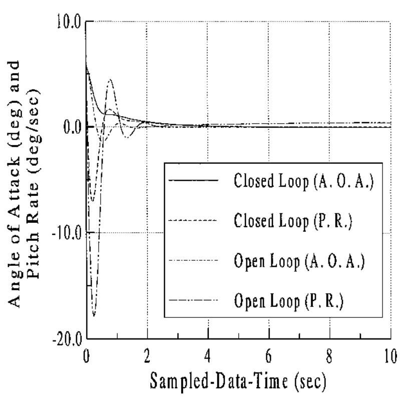

Fig. 1 Initial responses of angles of attack (AOA) and pitch rates (PR) with and without control.

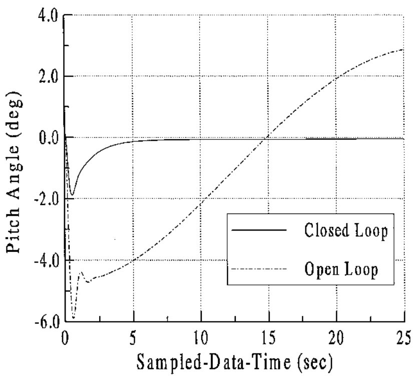

Fig. 2 Initial responses of pitch angles with and without control.

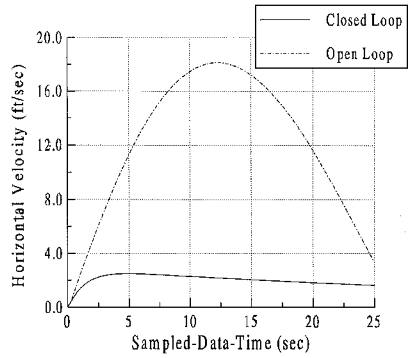

Fig. 3 Initial responses of horizontal velocities with and without control.

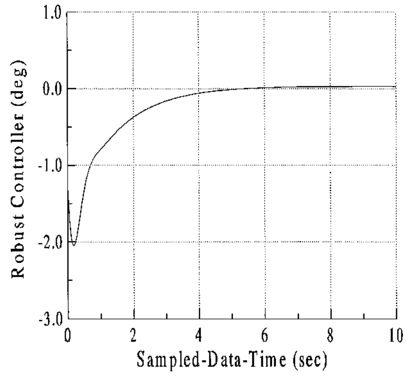

Fig. 4 Sampled-data time history of the robust controller.

Conclusions

This Note presents the development of a state feedback robust control technique for a class of discrete-time nonlinear uncertain systems. It is shown that the problem of robust H_∞ control for the nonlinear uncertain systems can be solved via a Riccati inequality for the corresponding linear uncertain systems. A linear dynamic state feedback controller has been designed based on the solution of the Riccati inequality.

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